A new way to iterate Brzeziński crossed products

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Abstract

If $A\otimes_{R,\sigma}V$ and $A\otimes_{P,\nu}W$ are two Brzeziński crossed products and $Q:W\otimes V\rightarrow V\otimes W$ is a linear map satisfying certain properties, we construct a Brzeziński crossed product $A\otimes_{S,\theta}(V\otimes W)$. This construction contains as a particular case the iterated twisted tensor product of algebras.

Introduction

The *twisted tensor product* of the associative unital algebras A and B is a new associative unital algebra structure built on the linear space $A\otimes B$ with the help of a linear map $R:B\otimes A\rightarrow A\otimes B$ called a *twisting map*. This construction, denoted by $A\otimes_R B$, appeared in several contexts and has various applications ([4], [10]). Concrete examples come especially from Hopf algebra theory, like for instance the smash product.

It was proved in [7] that twisted tensor products of algebras may be iterated. Namely, if $A\otimes_{R_1}B$, $B\otimes_{R_2}C$ and $A\otimes_{R_3}C$ are twisted tensor products and the twisting maps R_1, R_2, R_3 satisfy the braid relation $(id_A\otimes R_2)\circ(R_3\otimes id_B)\circ(id_C\otimes R_1)=(R_1\otimes id_C)\circ(id_B\otimes R_3)\circ(R_2\otimes id_A)$, then one can define certain twisted tensor products $A\otimes_{T_2}(B\otimes_{R_2}C)$ and $(A\otimes_{R_1}B)\otimes_{T_1}C$ that are equal as algebras (and this algebra is called the iterated twisted tensor product).

The Brzeziński crossed product, introduced in [1], is a common generalization of twisted tensor products of algebras and the Hopf crossed product (containing also as a particular case the quasi-Hopf smash product introduced in [3]). If A is an associative unital algebra, V is a linear space endowed with a distinguished element 1_V and $\sigma:V\otimes V\rightarrow A\otimes V$ and $R:V\otimes A\rightarrow A\otimes V$ are linear maps satisfying certain conditions, then the Brzeziński crossed product is a certain associative unital algebra structure on $A\otimes V$, denoted by $A\otimes_{R,\sigma}V$.

In [9] was proved that Brzeziński crossed products may be iterated, in the following sense. One can define first a "mirror version" of the Brzeziński crossed product, denoted by $W\overline{\otimes}_{P,\nu}D$

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(where D is an associative unital algebra, W is a linear space and P, ν are certain linear maps). Examples are twisted tensor products of algebras and the quasi-Hopf smash product introduced in [2]. Then it was proved that, if $W\overline{\otimes}_{P,\nu}D$ and $D\otimes_{R,\sigma}V$ are two Brzeziński crossed products and $Q: V\otimes W\rightarrow W\otimes D\otimes V$ is a linear map satisfying some conditions, then one can define certain Brzeziński crossed products $(W\overline{\otimes}_{P,\nu}D)\otimes_{\overline{R},\overline{\sigma}}V$ and $W\overline{\otimes}_{\overline{P},\overline{\nu}}(D\otimes_{R,\sigma}V)$ that are equal as algebras. Iterated twisted tensor products of algebras appear as a particular case of this construction, as well as the so-called quasi-Hopf two-sided smash product $A\#H\#B$ from [3].

The aim of this paper is to show that Brzeziński crossed products may be iterated in a different way, that will also contain as a particular case the iterated twisted tensor product of algebras. Namely, we prove that, if $A\otimes_{R,\sigma}V$ and $A\otimes_{P,\nu}W$ are two Brzeziński crossed products and $Q: W\otimes V\rightarrow V\otimes W$ is a linear map satisfying certain properties, then we can define two Brzeziński crossed products $A\otimes_{S,\theta}(V\otimes W)$ and $(A\otimes_{R,\sigma}V)\otimes_{T,\eta}W$ that are equal as algebras.

Our inspiration for looking at this new way of iterating Brzeziński crossed products came from the following result in graded ring theory: If G is a group, R is a G-graded ring, A and B are two finite left G-sets, then there exists a ring isomorphism between the smash products $R\#(A\times B)$ and $(R\#A)\#B$. This result was obtained in [5, Corollary 3.2], and it is useful in the study of the von Neumann regularity of rings of the type $R\#A$, cf. [5]. The smash product $R\#A$ of the $G-$graded ring R by a (finite) left $G-$set A was introduced in the paper [8] and it is a particular case of a more general construction. If H is a Hopf algebra, R an $H-$comodule algebra and C an $H-$module coalgebra, then we may consider the category ${}_R^C\mathcal{M}(H)$ of Doi-Koppinen Hopf modules (i.e. left $R-$modules and left $C-$comodules which satisfy certain compatibility relations). Then, the smash product $R\#A$ used in [5] is a particular smash product and it is the first example in the category ${}_R^C\mathcal{M}(H)$ (in the case when H is the groupring $k[G]$, R a $G-$graded ring and C the grouplike coalgebra $k[A]$ on a $G-$set A).

1 Preliminaries

We work over a commutative field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. By "algebra" we always mean an associative unital algebra. The multiplication of an algebra A is denoted by μ_A or simply μ when there is no danger of confusion, and we usually denote $\mu_A(a\otimes a')=aa'$ for all $a,a'\in A$. The unit of an algebra A is denoted by 1_A or simply 1 when there is no danger of confusion.

We recall from [4], [10] that, given two algebras A, B and a k-linear map $R: B\otimes A\rightarrow A\otimes B$, with Sweedler-type notation $R(b\otimes a)=a_R\otimes b_R$, for $a\in A$, $b\in B$, satisfying the conditions $a_R\otimes 1_R=a\otimes 1$, $1_R\otimes b_R=1\otimes b$, $(aa')_R\otimes b_R=a_Ra'_r\otimes (b_R)_r$, $a_R\otimes (bb')_R=(a_R)_r\otimes b_rb'_R$, for all $a,a'\in A$ and $b,b'\in B$ (where r and R are two different indices), if we define on $A\otimes B$ a new multiplication, by $(a\otimes b)(a'\otimes b')=aa'_R\otimes b_Rb'$, then this multiplication is associative with unit $1\otimes 1$. In this case, the map R is called a *twisting map* between A and B and the new algebra structure on $A\otimes B$ is denoted by $A\otimes_R B$ and called the *twisted tensor product* of A and B afforded by the map R.

We recall from [1] the construction of Brzeziński's crossed product:

Proposition 1.1 *([1]) Let $(A,\mu,1_A)$ be an (associative unital) algebra and V a vector space equipped with a distinguished element $1_V\in V$. Then the vector space $A\otimes V$ is an associative algebra with unit $1_A\otimes 1_V$ and whose multiplication has the property that $(a\otimes 1_V)(b\otimes v)=ab\otimes v$, for all $a,b\in A$ and $v\in V$, if and only if there exist linear maps $\sigma: V\otimes V\rightarrow A\otimes V$ and*

$R : V \otimes A \to A \otimes V$ satisfying the following conditions:

$$R(1_V \otimes a) = a \otimes 1_V, \quad R(v \otimes 1_A) = 1_A \otimes v, \quad \forall\, a \in A,\ v \in V, \tag{1.1}$$

$$\sigma(1_V \otimes v) = \sigma(v \otimes 1_V) = 1_A \otimes v, \quad \forall\, v \in V, \tag{1.2}$$

$$R \circ (id_V \otimes \mu) = (\mu \otimes id_V) \circ (id_A \otimes R) \circ (R \otimes id_A), \tag{1.3}$$

$$\begin{aligned}&(\mu \otimes id_V) \circ (id_A \otimes \sigma) \circ (R \otimes id_V) \circ (id_V \otimes \sigma)\\ &\qquad = (\mu \otimes id_V) \circ (id_A \otimes \sigma) \circ (\sigma \otimes id_V),\end{aligned} \tag{1.4}$$

$$\begin{aligned}&(\mu \otimes id_V) \circ (id_A \otimes \sigma) \circ (R \otimes id_V) \circ (id_V \otimes R)\\ &\qquad = (\mu \otimes id_V) \circ (id_A \otimes R) \circ (\sigma \otimes id_A).\end{aligned} \tag{1.5}$$

If this is the case, the multiplication of $A \otimes V$ is given explicitly by

$$\mu_{A \otimes V} = (\mu_2 \otimes id_V) \circ (id_A \otimes id_A \otimes \sigma) \circ (id_A \otimes R \otimes id_V),$$

where $\mu_2 = \mu \circ (id_A \otimes \mu) = \mu \circ (\mu \otimes id_A)$. We denote by $A \otimes_{R,\sigma} V$ this algebra structure and call it the crossed product *(or Brzeziński crossed product) afforded by the data (A, V, R, σ).*

If $A \otimes_{R,\sigma} V$ is a crossed product, we introduce the following Sweedler-type notation:

$$\begin{aligned}&R : V \otimes A \to A \otimes V, \quad R(v \otimes a) = a_R \otimes v_R,\\ &\sigma : V \otimes V \to A \otimes V, \quad \sigma(v \otimes v') = \sigma_1(v, v') \otimes \sigma_2(v, v'),\end{aligned}$$

for all $v, v' \in V$ and $a \in A$. With this notation, the multiplication of $A \otimes_{R,\sigma} V$ reads

$$(a \otimes v)(a' \otimes v') = aa'_R\sigma_1(v_R, v') \otimes \sigma_2(v_R, v'), \quad \forall\, a, a' \in A,\ v, v' \in V.$$

A twisted tensor product is a particular case of a crossed product (cf. [6]), namely, if $A \otimes_R B$ is a twisted tensor product of algebras then $A \otimes_R B = A \otimes_{R,\sigma} B$, where $\sigma : B \otimes B \to A \otimes B$ is given by $\sigma(b \otimes b') = 1_A \otimes bb'$, for all $b, b' \in B$.

Remark 1.2 *The conditions (1.3), (1.4) and (1.5) for R, σ may be written in Sweedler-type notation respectively as*

$$(aa')_R \otimes v_R = a_R a'_r \otimes (v_R)_r, \tag{1.6}$$

$$\begin{aligned}&\sigma_1(y, z)_R \sigma_1(x_R, \sigma_2(y, z)) \otimes \sigma_2(x_R, \sigma_2(y, z))\\ &\qquad = \sigma_1(x, y)\sigma_1(\sigma_2(x, y), z) \otimes \sigma_2(\sigma_2(x, y), z),\end{aligned} \tag{1.7}$$

$$(a_R)_r \sigma_1(v_r, v'_R) \otimes \sigma_2(v_r, v'_R) = \sigma_1(v, v')a_R \otimes \sigma_2(v, v')_R, \tag{1.8}$$

for all $a, a' \in A$ and $x, y, z, v, v' \in V$, where r is another copy of R.

2 The main result and examples

Theorem 2.1 *Let $A \otimes_{R,\sigma} V$ and $A \otimes_{P,\nu} W$ be two crossed products and $Q : W \otimes V \to V \otimes W$ a linear map, with notation $Q(w \otimes v) = v_Q \otimes w_Q$, for all $v \in V$ and $w \in W$. Assume that the following conditions are satisfied:*

(i) Q is unital, in the sense that

$$Q(1_W \otimes v) = v \otimes 1_W, \quad Q(w \otimes 1_V) = 1_V \otimes w, \quad \forall\, v \in V,\ w \in W. \tag{2.1}$$

(ii) the braid relation for R, P, Q, i.e.

$$(id_A \otimes Q) \circ (P \otimes id_V) \circ (id_W \otimes R) = (R \otimes id_W) \circ (id_V \otimes P) \circ (Q \otimes id_A), \tag{2.2}$$

or, equivalently,

$$(a_R)_P \otimes (v_R)_Q \otimes (w_P)_Q = (a_P)_R \otimes (v_Q)_R \otimes (w_Q)_P, \quad \forall\, a \in A,\ v \in V,\ w \in W. \tag{2.3}$$

(iii) we have the following hexagonal relation between σ, P, Q:

$$(id_A \otimes Q) \circ (P \otimes id_V) \circ (id_W \otimes \sigma) = (\sigma \otimes id_W) \circ (id_V \otimes Q) \circ (Q \otimes id_V), \tag{2.4}$$

or, equivalently,

$$\sigma_1(v, v')_P \otimes \sigma_2(v, v')_Q \otimes (w_P)_Q = \sigma_1(v_Q, v'_q) \otimes \sigma_2(v_Q, v'_q) \otimes (w_Q)_q, \tag{2.5}$$

for all $v, v' \in V$ and $w \in W$, where q is another copy of Q.

(iv) we have the following hexagonal relation between ν, R, Q:

$$(id_A \otimes Q) \circ (\nu \otimes id_V) = (R \otimes id_W) \circ (id_V \otimes \nu) \circ (Q \otimes id_W) \circ (id_W \otimes Q), \tag{2.6}$$

or, equivalently,

$$\nu_1(w, w') \otimes v_Q \otimes \nu_2(w, w')_Q = \nu_1(w_q, w'_Q)_R \otimes ((v_Q)_q)_R \otimes \nu_2(w_q, w'_Q), \tag{2.7}$$

for all $v \in V$ and $w, w' \in W$, where q is another copy of Q.

Define the linear maps

$$\begin{aligned}
&S : (V \otimes W) \otimes A \to A \otimes (V \otimes W), \quad S := (R \otimes id_W) \circ (id_V \otimes P),\\
&\theta : (V \otimes W) \otimes (V \otimes W) \to A \otimes (V \otimes W),\\
&\theta := (\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes R \otimes id_W) \circ (\sigma \otimes \nu) \circ (id_V \otimes Q \otimes id_W),\\
&T : W \otimes (A \otimes V) \to (A \otimes V) \otimes W, \quad T := (id_A \otimes Q) \circ (P \otimes id_V),\\
&\eta : W \otimes W \to (A \otimes V) \otimes W,\\
&\eta(w \otimes w') = (\nu_1(w, w') \otimes 1_V) \otimes \nu_2(w, w'), \quad \forall\, w, w' \in W.
\end{aligned}$$

Then we have a crossed product $A \otimes_{S,\theta} (V \otimes W)$ (with respect to $1_{V\otimes W} := 1_V \otimes 1_W$), we have a crossed product $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ and we have an algebra isomorphism $A \otimes_{S,\theta} (V \otimes W) \simeq (A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ given by the trivial identification.

Proof. We prove first that $A \otimes_{S,\theta} (V \otimes W)$ is a crossed product, i.e. we need to prove the relations (1.1)-(1.5) with R replaced by S, σ replaced by θ etc. The relations (1.1) and (1.2) follow immediately by (2.1) and the relations (1.1) and (1.2) for R, σ and P, ν. Note that the maps S and θ are defined explicitely by

$$\begin{aligned}
&S(v \otimes w \otimes a) = (a_P)_R \otimes v_R \otimes w_P,\\
&\theta(v \otimes w \otimes v' \otimes w') = \sigma_1(v, v'_Q)\nu_1(w_Q, w')_R \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2(w_Q, w'),
\end{aligned}$$

for all $v, v' \in V$, $w, w' \in W$ and $a \in A$. We will denote by $R = r = \mathcal{R} = \overline{R}$, $Q = q = \overline{Q}$ and $P = p$ some more copies of R, Q and P.

<u>Proof of (1.3)</u>:

$S\circ(id_V\otimes id_W\otimes\mu_A)(v\otimes w\otimes a\otimes a')$

$$\begin{aligned}
&= S(v\otimes w\otimes aa')\\
&= ((aa')_P)_R\otimes v_R\otimes w_P\\
&\overset{(1.6)}{=} (a_Pa'_p)_R\otimes v_R\otimes (w_P)_p\\
&\overset{(1.6)}{=} (a_P)_R(a'_p)_r\otimes (v_R)_r\otimes (w_P)_p\\
&= (\mu_A\otimes id_V\otimes id_W)((a_P)_R\otimes (a'_p)_r\otimes (v_R)_r\otimes (w_P)_p)\\
&= (\mu_A\otimes id_V\otimes id_W)\circ(id_A\otimes S)((a_P)_R\otimes v_R\otimes w_P\otimes a')\\
&= (\mu_A\otimes id_V\otimes id_W)\circ(id_A\otimes S)\circ(S\otimes id_A)(v\otimes w\otimes a\otimes a'),\quad q.e.d.
\end{aligned}$$

Proof of (1.4):

$(\mu_A\otimes id_V\otimes id_W)\circ(id_A\otimes\theta)\circ(S\otimes id_V\otimes id_W)\circ(id_V\otimes id_W\otimes\theta)(v\otimes w\otimes v'\otimes w'\otimes v''\otimes w'')$

$$\begin{aligned}
&= (\mu_A\otimes id_V\otimes id_W)\circ(id_A\otimes\theta)\circ(S\otimes id_V\otimes id_W)(v\otimes w\otimes \sigma_1(v',v''_Q)\nu_1(w'_Q,w'')_R\\
&\quad\otimes\sigma_2(v',v''_Q)_R\otimes\nu_2(w'_Q,w''))\\
&= (\mu_A\otimes id_V\otimes id_W)\circ(id_A\otimes\theta)(([\sigma_1(v',v''_Q)\nu_1(w'_Q,w'')_R]_P)_r\otimes v_r\otimes w_P\\
&\quad\otimes\sigma_2(v',v''_Q)_R\otimes\nu_2(w'_Q,w''))\\
&= ([\sigma_1(v',v''_Q)\nu_1(w'_Q,w'')_R]_P)_r\sigma_1(v_r,(\sigma_2(v',v''_Q)_R)_q)\nu_1((w_P)_q,\nu_2(w'_Q,w''))_{\mathcal{R}}\\
&\quad\otimes\sigma_2(v_r,(\sigma_2(v',v''_Q)_R)_q)_{\mathcal{R}}\otimes\nu_2((w_P)_q,\nu_2(w'_Q,w''))\\
&\overset{(1.6)}{=} (\sigma_1(v',v''_Q)_P(\nu_1(w'_Q,w'')_R)_p)_r\sigma_1(v_r,(\sigma_2(v',v''_Q)_R)_q)\nu_1(((w_P)_p)_q,\nu_2(w'_Q,w''))_{\mathcal{R}}\\
&\quad\otimes\sigma_2(v_r,(\sigma_2(v',v''_Q)_R)_q)_{\mathcal{R}}\otimes\nu_2(((w_P)_p)_q,\nu_2(w'_Q,w''))\\
&\overset{(1.6)}{=} (\sigma_1(v',v''_Q)_P)_{\overline{R}}((\nu_1(w'_Q,w'')_R)_p)_r\sigma_1((v_{\overline{R}})_r,(\sigma_2(v',v''_Q)_R)_q)\nu_1(((w_P)_p)_q,\nu_2(w'_Q,w''))_{\mathcal{R}}\\
&\quad\otimes\sigma_2((v_{\overline{R}})_r,(\sigma_2(v',v''_Q)_R)_q)_{\mathcal{R}}\otimes\nu_2(((w_P)_p)_q,\nu_2(w'_Q,w''))\\
&\overset{(2.3)}{=} (\sigma_1(v',v''_Q)_P)_{\overline{R}}((\nu_1(w'_Q,w'')_p)_R)_r\sigma_1((v_{\overline{R}})_r,(\sigma_2(v',v''_Q)_q)_R)\nu_1(((w_P)_q)_p,\nu_2(w'_Q,w''))_{\mathcal{R}}\\
&\quad\otimes\sigma_2((v_{\overline{R}})_r,(\sigma_2(v',v''_Q)_q)_R)_{\mathcal{R}}\otimes\nu_2(((w_P)_q)_p,\nu_2(w'_Q,w''))\\
&\overset{(1.8)}{=} (\sigma_1(v',v''_Q)_P)_{\overline{R}}\sigma_1(v_{\overline{R}},\sigma_2(v',v''_Q)_q)(\nu_1(w'_Q,w'')_p)_R\nu_1(((w_P)_q)_p,\nu_2(w'_Q,w''))_{\mathcal{R}}\\
&\quad\otimes(\sigma_2(v_{\overline{R}},\sigma_2(v',v''_Q)_q)_R)_{\mathcal{R}}\otimes\nu_2(((w_P)_q)_p,\nu_2(w'_Q,w''))\\
&\overset{(2.5)}{=} \sigma_1(v'_{\overline{Q}},(v''_Q)_q)_{\overline{R}}\sigma_1(v_{\overline{R}},\sigma_2(v'_{\overline{Q}},(v''_Q)_q))(\nu_1(w'_Q,w'')_p)_R\nu_1(((w_{\overline{Q}})_q)_p,\nu_2(w'_Q,w''))_{\mathcal{R}}\\
&\quad\otimes(\sigma_2(v_{\overline{R}},\sigma_2(v'_{\overline{Q}},(v''_Q)_q))_R)_{\mathcal{R}}\otimes\nu_2(((w_{\overline{Q}})_q)_p,\nu_2(w'_Q,w''))\\
&\overset{(1.7)}{=} \sigma_1(v,v'_{\overline{Q}})\sigma_1(\sigma_2(v,v'_{\overline{Q}}),(v''_Q)_q)(\nu_1(w'_Q,w'')_p)_R\nu_1(((w_{\overline{Q}})_q)_p,\nu_2(w'_Q,w''))_{\mathcal{R}}\\
&\quad\otimes(\sigma_2(\sigma_2(v,v'_{\overline{Q}}),(v''_Q)_q)_R)_{\mathcal{R}}\otimes\nu_2(((w_{\overline{Q}})_q)_p,\nu_2(w'_Q,w''))\\
&\overset{(1.6)}{=} \sigma_1(v,v'_{\overline{Q}})\sigma_1(\sigma_2(v,v'_{\overline{Q}}),(v''_Q)_q)[\nu_1(w'_Q,w'')_p\nu_1(((w_{\overline{Q}})_q)_p,\nu_2(w'_Q,w''))]_R\\
&\quad\otimes\sigma_2(\sigma_2(v,v'_{\overline{Q}}),(v''_Q)_q)_R\otimes\nu_2(((w_{\overline{Q}})_q)_p,\nu_2(w'_Q,w''))\\
&\overset{(1.7)}{=} \sigma_1(v,v'_{\overline{Q}})\sigma_1(\sigma_2(v,v'_{\overline{Q}}),(v''_Q)_q)[\nu_1((w_{\overline{Q}})_q,w'_Q)\nu_1(\nu_2((w_{\overline{Q}})_q,w'_Q),w'')]_R\\
&\quad\otimes\sigma_2(\sigma_2(v,v'_{\overline{Q}}),(v''_Q)_q)_R\otimes\nu_2(\nu_2((w_{\overline{Q}})_q,w'_Q),w'')
\end{aligned}$$

$$
\begin{aligned}
&\overset{(1.8)}{=} && \sigma_1(v, v'_{\overline{Q}})\{[\nu_1((w_{\overline{Q}})_q, w'_Q)\nu_1(\nu_2((w_{\overline{Q}})_q, w'_Q), w'')]_R\}_r\sigma_1(\sigma_2(v, v'_{\overline{Q}})_r, ((v''_Q)_q)_R) \\
& && \otimes\sigma_2(\sigma_2(v, v'_{\overline{Q}})_r, ((v''_Q)_q)_R) \otimes \nu_2(\nu_2((w_{\overline{Q}})_q, w'_Q), w'') \\
&\overset{(1.6)}{=} && \sigma_1(v, v'_{\overline{Q}})[\nu_1((w_{\overline{Q}})_q, w'_Q)_R\nu_1(\nu_2((w_{\overline{Q}})_q, w'_Q), w'')_{\mathcal{R}}]_r\sigma_1(\sigma_2(v, v'_{\overline{Q}})_r, (((v''_Q)_q)_R)_{\mathcal{R}}) \\
& && \otimes\sigma_2(\sigma_2(v, v'_{\overline{Q}})_r, (((v''_Q)_q)_R)_{\mathcal{R}}) \otimes \nu_2(\nu_2((w_{\overline{Q}})_q, w'_Q), w'') \\
&\overset{(2.7)}{=} && \sigma_1(v, v'_{\overline{Q}})[\nu_1(w_{\overline{Q}}, w')\nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_{\mathcal{R}}]_r\sigma_1(\sigma_2(v, v'_{\overline{Q}})_r, (v''_Q)_{\mathcal{R}}) \\
& && \otimes\sigma_2(\sigma_2(v, v'_{\overline{Q}})_r, (v''_Q)_{\mathcal{R}}) \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'') \\
&\overset{(1.6)}{=} && \sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R(\nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_{\mathcal{R}})_r\sigma_1((\sigma_2(v, v'_{\overline{Q}})_R)_r, (v''_Q)_{\mathcal{R}}) \\
& && \otimes\sigma_2((\sigma_2(v, v'_{\overline{Q}})_R)_r, (v''_Q)_{\mathcal{R}}) \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'') \\
&\overset{(1.8)}{=} && \sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R\sigma_1(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q)\nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_r \\
& && \otimes\sigma_2(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q)_r \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'') \\
&= && (\mu_A \otimes id_V \otimes id_W)(\sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R \otimes \sigma_1(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q)\nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_r \\
& && \otimes\sigma_2(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q)_r \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'')) \\
&= && (\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes \theta)(\sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R \\
& && \otimes\sigma_2(v, v'_{\overline{Q}})_R \otimes \nu_2(w_{\overline{Q}}, w') \otimes v'' \otimes w'') \\
&= && (\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes \theta) \circ (\theta \otimes id_V \otimes id_W)(v \otimes w \otimes v' \otimes w' \otimes v'' \otimes w''), \quad q.e.d.
\end{aligned}
$$

Proof of (1.5):

$$
\begin{aligned}
&(\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes \theta) \circ (S \otimes id_V \otimes id_W) \circ (id_V \otimes id_W \otimes S)(v \otimes w \otimes v' \otimes w' \otimes a) \\
&= (\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes \theta) \circ (S \otimes id_V \otimes id_W)(v \otimes w \otimes (a_P)_R \otimes v'_R \otimes w'_P) \\
&= (\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes \theta)((((a_P)_R)_p)_r \otimes v_r \otimes w_p \otimes v'_R \otimes w'_P) \\
&= (((a_P)_R)_p)_r\sigma_1(v_r, (v'_R)_Q)\nu_1((w_p)_Q, w'_P)_{\mathcal{R}} \otimes \sigma_2(v_r, (v'_R)_Q)_{\mathcal{R}} \otimes \nu_2((w_p)_Q, w'_P) \\
&\overset{(2.3)}{=} (((a_P)_p)_R)_r\sigma_1(v_r, (v'_Q)_R)\nu_1((w_Q)_p, w'_P)_{\mathcal{R}} \otimes \sigma_2(v_r, (v'_Q)_R)_{\mathcal{R}} \otimes \nu_2((w_Q)_p, w'_P) \\
&\overset{(1.8)}{=} \sigma_1(v, v'_Q)((a_P)_p)_R\nu_1((w_Q)_p, w'_P)_{\mathcal{R}} \otimes (\sigma_2(v, v'_Q)_R)_{\mathcal{R}} \otimes \nu_2((w_Q)_p, w'_P) \\
&\overset{(1.6)}{=} \sigma_1(v, v'_Q)[(a_P)_p\nu_1((w_Q)_p, w'_P)]_R \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2((w_Q)_p, w'_P) \\
&\overset{(1.8)}{=} \sigma_1(v, v'_Q)[\nu_1(w_Q, w')a_P]_R \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2(w_Q, w')_P \\
&\overset{(1.6)}{=} \sigma_1(v, v'_Q)\nu_1(w_Q, w')_R(a_P)_r \otimes (\sigma_2(v, v'_Q)_R)_r \otimes \nu_2(w_Q, w')_P \\
&= (\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes S)(\sigma_1(v, v'_Q)\nu_1(w_Q, w')_R \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2(w_Q, w') \otimes a) \\
&= (\mu_A \otimes id_V \otimes id_W) \circ (id_A \otimes S) \circ (\theta \otimes id_A)(v \otimes w \otimes v' \otimes w' \otimes a),
\end{aligned}
$$

so $A \otimes_{S,\theta} (V \otimes W)$ is indeed a crossed product. With a similar computation one can prove that $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ is a crossed product, so the only thing left to prove is that the multiplications of $A \otimes_{S,\theta} (V \otimes W)$ and $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ coincide. A straightforward computation shows that the multiplication of $A \otimes_{S,\theta} (V \otimes W)$ is given by the formula

$$
(a \otimes v \otimes w)(a' \otimes v' \otimes w') = a(a'_P)_{\mathcal{R}}\sigma_1(v_{\mathcal{R}}, v'_Q)\nu_1((w_P)_Q, w')_r \otimes \sigma_2(v_{\mathcal{R}}, v'_Q)_r \otimes \nu_2((w_P)_Q, w').
$$

We compute now the multiplication of $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$:

$$
\begin{aligned}
&(a \otimes v \otimes w)(a' \otimes v' \otimes w') \\
&\quad = (a \otimes v)(a' \otimes v')_T \eta_1(w_T, w') \otimes \eta_2(w_T, w') \\
&\quad = (a \otimes v)(a'_P \otimes v'_Q)\eta_1((w_P)_Q, w') \otimes \eta_2((w_P)_Q, w') \\
&\quad = (a \otimes v)(a'_P \otimes v'_Q)(\nu_1((w_P)_Q, w') \otimes 1_V) \otimes \nu_2((w_P)_Q, w') \\
&\quad = (a \otimes v)(a'_P \nu_1((w_P)_Q, w')_R \otimes (v'_Q)_R) \otimes \nu_2((w_P)_Q, w') \\
&\quad = a[a'_P \nu_1((w_P)_Q, w')_R]_r \sigma_1(v_r, (v'_Q)_R) \otimes \sigma_2(v_r, (v'_Q)_R) \otimes \nu_2((w_P)_Q, w') \\
&\quad \overset{(1.6)}{=} a(a'_P)_{\mathcal{R}}(\nu_1((w_P)_Q, w')_R)_r \sigma_1((v_{\mathcal{R}})_r, (v'_Q)_R) \otimes \sigma_2((v_{\mathcal{R}})_r, (v'_Q)_R) \otimes \nu_2((w_P)_Q, w') \\
&\quad \overset{(1.8)}{=} a(a'_P)_{\mathcal{R}} \sigma_1(v_{\mathcal{R}}, v'_Q)\nu_1((w_P)_Q, w')_r \otimes \sigma_2(v_{\mathcal{R}}, v'_Q)_r \otimes \nu_2((w_P)_Q, w'),
\end{aligned}
$$

and we can see that the two multiplications coincide. $\square$

Example 2.2 We recall from [7] what was called there an iterated twisted tensor product of algebras. Let A, B, C be associative unital algebras, $R_1 : B \otimes A \to A \otimes B$, $R_2 : C \otimes B \to B \otimes C$, $R_3 : C \otimes A \to A \otimes C$ twisting maps satisfying the braid equation

$$(id_A \otimes R_2) \circ (R_3 \otimes id_B) \circ (id_C \otimes R_1) = (R_1 \otimes id_C) \circ (id_B \otimes R_3) \circ (R_2 \otimes id_A).$$

Then we have an algebra structure on $A \otimes B \otimes C$ (called the iterated twisted tensor product) with unit $1_A \otimes 1_B \otimes 1_C$ and multiplication

$$(a \otimes b \otimes c)(a' \otimes b' \otimes c') = a(a'_{R_3})_{R_1} \otimes b_{R_1} b'_{R_2} \otimes (c_{R_3})_{R_2} c'.$$

We define $V = B$, $W = C$, $R = R_1$, $P = R_3$, $Q = R_2$ and the linear maps

$$
\begin{aligned}
&\sigma : V \otimes V \to A \otimes V, \quad \sigma(b \otimes b') = 1_A \otimes bb', \quad \forall\, b, b' \in V, \\
&\nu : W \otimes W \to A \otimes W, \quad \nu(c \otimes c') = 1_A \otimes cc', \quad \forall\, c, c' \in W.
\end{aligned}
$$

Then, for the crossed products $A \otimes_{R,\sigma} V = A \otimes_{R_1} B$, $A \otimes_{P,\nu} W = A \otimes_{R_3} C$ and the map Q, one can check that the hypotheses of Theorem 2.1 are satisfied and the crossed products $A \otimes_{S,\theta} (V \otimes W) \equiv (A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ (notation as in Theorem 2.1) coincide with the iterated twisted tensor product.

Example 2.3 *Let $A \otimes_{R,\sigma} V$ be a crossed product and W an (associative unital) algebra. Define the linear maps*

$$
\begin{aligned}
&P : W \otimes A \to A \otimes W, \quad P(w \otimes a) = a \otimes w, \quad \forall\, a \in A,\ w \in W, \\
&\nu : W \otimes W \to A \otimes W, \quad \nu(w \otimes w') = 1_A \otimes ww', \quad \forall\, w, w' \in W,
\end{aligned}
$$

so we have the crossed product $A \otimes_{P,\nu} W$ which is just the ordinary tensor product of algebras $A \otimes W$. Define the linear map $Q : W \otimes V \to V \otimes W$, $Q(w \otimes v) = v \otimes w$, for all $v \in V$, $w \in W$. Then one can easily check that the hypotheses of Theorem 2.1 are satisfied.

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